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COMMENT

Normal ordering formulae for some boson operators

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Abstract. The normal ordering formula for $(N+a^r)^n$, where $N=a^+a$, is derived for arbitrary r and n. This is a generalisation of the results recently obtained by Mikhailov, for $r=1, 2$. Some other closely related results are also discussed.

Mikhailov (1983) has recently derived normal ordering formulae for the boson operators $(N+a)^n$, $(N+a^2)^n$ as well as $(a^++a^r)^n$. His approach is based on the definition of symmetrisers which are simply related to the operators of interest and which satisfy certain recurrence relations. The latter enabled him to derive recurrence relations for the coefficients in the normal ordering formulae, the use of which provided the final closed form expressions for these coefficients.

In the present note we obtain the normal ordering formula for $(N+a^r)^n$, for arbitrary integral r (and n). The approach is similar to that used in a closely related context by Witschel (1975).

We first point out that

$$[N, a^r] = -ra^r \tag{1}$$

so that equation (B7) of Kirzhnits (1967) is applicable, resulting in the identity

$$\exp[\alpha(N+a^r)] = e^{\alpha N} \exp[r^{-1}(e^{r\alpha}-1)a^r]. \tag{2}$$

We shall now use the identity (Louisell 1964, p 116)

$$e^{\alpha N} = \sum_{i=0}^{\infty} \frac{1}{i!}(e^{\alpha}-1)^i (a^+)^i a^i, \tag{3}$$

the expansion

$$\exp[r^{-1}(e^{r\alpha}-1)a^r] = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{1}{r}\right)^j (e^{r\alpha}-1)^j a^{rj}, \tag{4}$$

and the fact that $(e^{\alpha}-1)^i$ (as well as $(e^{r\alpha}-1)^j$) is a generating function for the Stirling numbers of the second kind (Abramowitz and Stegun 1965).

Equating coefficients of α^n in (2) we finally obtain

$$(N+a^r)^n = \sum_{i=0}^{n} \sum_{j=0}^{n-i} C_{i,j}^{(n,r)} (a^+)^i a^{i+rj} \tag{5}$$

where

$$C_{i,j}^{(n,r)} = \sum_{k=i}^{n-j} \binom{n}{k} S(k,i) S(n-k,j) r^{n-j-k}. \tag{6}$$

$S(n, m)$ are the Stirling numbers of the second kind.

Of the two special cases considered by Mikhailov, $r = 1$ is particularly interesting because of the identity

$$C_{i,j}^{(n,1)} = \binom{i+j}{i} S(n, i+j)$$

resulting from a recurrence relation satisfied by the Stirling numbers (Abramowitz and Stegun 1965, p 825).

This result, which is identical with that of Mikhailov (1983), can also be obtained more straightforwardly by noting that for $r = 1$, equation (2) becomes

$$e^{\alpha(N+a)} = e^{\alpha N} \exp[(e^{\alpha}-1)a] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(e^{\alpha}-1)^{i+j}}{i!j!} (a^+)^i a^{i+j}. \tag{7}$$

Expanding in powers of α and equating coefficients of α^n we obtain the desired result.

The other normal ordering formula considered by Mikhailov (1983), that of $(a^+ + a^r)^n$, can also be obtained using the above procedure. Here, however, the non-vanishing commutators are

$$[a^+, a^r] = -ra^{r-1},$$

$$[a^+, [a^+, a^r]] = r(r-1)a^{r-2},$$

$$\vdots$$

$$\underbrace{[a^+, [a^+, [\ldots [a^+}_{k \text{ times}}, a^r] \ldots]]] = (-1)^k \frac{r!}{(r-k)!} a^{r-k}.$$

Using equation (B5) of Kirzhnits (1967) we obtain

$$\exp[\alpha(a^+ + a^r)] = \exp(\alpha a^+) \exp\left[\sum_{i=0}^{r} \frac{1}{i+1} \binom{r}{i} \alpha^{i+1} a^{r-i}\right]. \tag{8}$$

Thus,

$$(a^+ + a^r)^n = n! \sum_{j=0}^{\infty} \sum_{j_0=0}^{\infty} \sum_{j_1=0}^{\infty} \cdots \sum_{j_r=0}^{\infty} \frac{1}{j!j_0!j_1!\ldots j_r!} \prod_{i=0}^{r} \left[\frac{1}{i+1}\binom{r}{i}\right]^{j_i} (a^+)^j a^{\Sigma_{i=0}^r (r-i)j_i} \tag{9}$$

$$\left(j + \sum_{i=0}^{r} (i+1) j_i = n\right)$$

which can be shown to be identical with the corresponding result of Mikhailov. Clearly, the condition $j + \Sigma_{i=0}^r (i+1) j_i = n$ reduces the number of summations from $r+2$ to $r+1$ and also results in all the summations being finite. Note that equation (17) of Mikhailov involves r summations, and one more summation is necessary to transform the symmetrisers to the desired expression for $(a^+ + a^r)^n$.

A minor modification of the power series approach enables the derivation of normal ordering formulae for the symmetrisers. This follows from the fact that

$$e^{\alpha A+\beta B}=\sum_{i=0}^{\infty}\frac{1}{i!}(\alpha A+\beta B)^i=\sum_{i=0}^{\infty}\frac{1}{i!}\sum_{j=0}^{i}\alpha^j\beta^{i-j}\{A^j,B^{i-j}\} \tag{10}$$

so that $e^{\alpha A+\beta B}$ is a generating function for the symmetrisers $\{A^j, B^{i-j}\}$. In the derivation of this result equations (7) and (9) of Mikhailov (1983) were used.

For $A=N$, $B=a^r$ we have

$$\exp(\alpha N+\beta a^r)=e^{\alpha N}\exp\left(\frac{\beta}{\alpha r}(e^{\alpha r}-1)a^r\right) \tag{11}$$

so that expanding in powers of α and β and equating coefficients we obtain

$$\{N^{n-j},(a^r)^j\}=\sum_{i=0}^{n}C_{i,j}^{(n,r)}(a^+)^i a^{rj+i} \tag{12}$$

where $C_{i,j}^{(n,r)}$ is given by (6).

Thus, summing over j we obtain (5). Equation (12) is a generalisation to arbitrary r of equations (21)–(23) and (24)–(31) ($r=1$ and 2, respectively) of Mikhailov (1983).

The symmetrisers $\{(a^+)^{n-j},(a^r)^j\}$ can be obtained starting from the identity

$$\exp(\alpha a^+ +\beta a^r)=\exp(\alpha a^+)\exp\left[\beta\sum_{i=0}^{r}\frac{1}{i+1}\binom{r}{i}\alpha^i a^{r-i}\right]. \tag{13}$$

As this does not involve any generalisation of the corresponding result of Mikhailov, no further details are necessary.

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